SHIP WAVES IN UNIFORMLY ACCELERATED MOTION

(KORABEL'NYE VOLNY PRI RAVNOUSKORENNOM DVIZHENII)

PMM Vol.29, № 1, 1965, pp.192-196

R.A.GRUNTFEST

(Rostov-on-Don)

(Received April 21, 1964)

The present paper is concerned with the waves generated by a ship moving in an incompressible, viscous fluid of infinite depth. The motion is assumed rectilinear with uniform acceleration and with zero initial velocity, although the last restriction is not in principle necessary. In order to simplify the problem, the ship is replaced by a point impulse of pressures.

A solution was obtained in [1] for the waves generated by a surface pressure distribution $p_0(x, y, t)$ on the surface of a viscous, incompressible liquid of infinite depth initially at rest; the equation of the free surface was found to be

$$\zeta(x, y, t) = -\frac{1}{2\pi\mu}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(u, v, \tau)\, G(t-\tau, r)\, du\, dv\, d\tau \qquad (1)$$

$$G(r, t) \sim \frac{g\nu t^3}{4\sqrt{2}r^4}\exp\left(-\frac{\nu g^2 t^5}{8r^4}\right)\sin\frac{gt^2}{4r}, \qquad r = \sqrt{(x-u)^2+(y-v)^2}$$

The origin of coordinates is taken at the free surface in the equilibrium position, with the z-axis directed vertically upward.

Under the assumption that the ship moves in the negative x-direction, the expression for p_0 is found in the given case to be

$$p_0(x, y, t) = Q\delta(y)\,\delta(x + {}^1/_2 at^2)$$

where a is the acceleration of the ship, and Q is the constant intensity of the impulse. When this expression is substituted into (1), the equation of the free surface assumes the form

$$\zeta(x, y, t) \sim -\frac{Q}{2\pi\mu}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(v)\,\delta(u + {}^1/_2 a\tau^2)\;G(t-\tau, r)\, du\, dv\, d\tau$$

The properties of the δ-function permit this to be written in the form

$$\zeta(x, y, t) \sim -\frac{Qg}{8\sqrt{2}\pi\rho}\int_0^t\frac{(t-\tau)^3}{r^4}\exp\left(-\frac{\nu g^2(t-\tau)^5}{8r^4}\right)\sin\frac{g(t-\tau)^2}{4r}\,d\tau \qquad (2)$$

$$r = \sqrt{(x + {}^1/_2 a\tau^2)^2 + y^2}$$

A change of variables is useful at this point. Let $\tau' = t - \tau$, $x' = x + \frac{1}{2}at^2$, i.e. let the origin of coordinates be placed at the point impulse; at the same time, $t = 0$ corresponds to the origin of coordinates, while the time t

corresponds to the start of the motion. With the primes dropped from x and τ, we obtain

$$\zeta(x, y, t) \sim -\frac{Qg}{8\sqrt{2}\pi\rho}\int_0^t \frac{\tau^3}{r^4}\exp\left(-\frac{\nu g^2\tau^5}{8r^4}\right)\sin\frac{g\tau^2}{4r}\,d\tau \tag{3}$$

$$r = \sqrt{(x - at\tau + {}^1/_2 a\tau^2)^2 + y^2} \tag{4}$$

Preparatory to finding the integral in Equation (3) by the method of stationary phase, it is rewritten in the form

$$\zeta(x, y, t) \sim \int_0^t \psi(\tau)\sin\varphi(\tau)\,d\tau$$

$$\psi(\tau) = -\frac{Qg}{8\sqrt{2}\pi\rho}\frac{\tau^3}{r^4}\exp\left(-\frac{\nu g^2\tau^5}{8r^4}\right), \qquad \varphi(\tau) = \frac{g\tau^2}{4r}$$

Then

$$\frac{d\varphi}{d\tau} = \frac{g}{4}\left(\frac{2\tau}{r} - \frac{\tau^2}{r^2}\frac{dr}{d\tau}\right) \tag{5}$$

The condition of stationary phase leads to the following relationship

$$\frac{dr}{d\tau} = \frac{2r}{\tau} = a(t-\tau)\cos\theta \tag{6}$$

Here $dr/d\tau$ is found in accordance with Equation (4), and θ is the angle between the vector $\mathbf{r}$ and the negative x-direction (Fig.1). Expression (6) is more conveniently written in the form

$$r = {}^1/_2 a\tau(t-\tau)\cos\theta \tag{7}$$

The significance of this relationship lies in the fact that for a given point $P(x, y)$, it determines those points N_1 on the course of the ship that alone play a role in generating a disturbance at P and which, following Stoker, will be called influence points. Equation (7) may be interpreted also in a different way: it yields the polar coordinates r, θ relative to a pole at N_1 of all points for which N_1 is an influence point in the above sense. These points lie on a circle of diameter $\frac{1}{2}a\tau(t-\tau)$ and whose center lies on the ship's course at a distance $\frac{3}{4}at\tau - \frac{1}{4}a\tau^2$ from the origin of coordinates (Fig.1). It is now clear that the disturbance generated by the ship does not affect the entire liquid surface, but is limited to the region covered by the totality of circles of influence. The boundary of this region will be the envelope of a one-parameter family of circles of influence. The equation of the family of these circles is

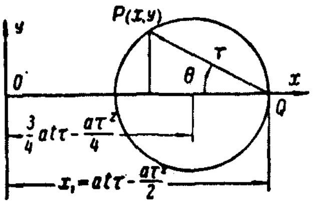

Fig. 1

$$(x - {}^3/_4 a\tau t + {}^1/_4 a\tau^2)^2 + y^2 = {}^1/_{16}\, a^2\tau^2(t-\tau)^2 \qquad (0 \leqslant \tau \leqslant t) \tag{8}$$

Differentiation with respect to the parameter τ yields

$$(x - {}^3/_4 a\tau t + {}^1/_4 a\tau^2)(a\tau - {}^3/_2 at) = {}^1/_8 a^2\tau(t-\tau)(t-2\tau) \tag{9}$$

From this it follows that

$$x = at\tau\frac{4t-3\tau}{6t-4\tau}, \qquad y^2 = \frac{a^2\tau^2(t-\tau)^2 t}{2(3t-2\tau)^2}$$

The parameter τ, having the dimensions of time, can be replaced by a dimensionless parameter α, defined by the relationship $\tau = \alpha t$, $0 \leqslant \alpha \leqslant 1$.

The equation defining the boundary of the region of influence is then

$$x = \frac{at^2}{2}\,\alpha\,\frac{4-3\alpha}{3-2\alpha}, \qquad y = \pm\frac{at^2}{2}\,\frac{\sqrt{2\alpha}\,(1-\alpha)^{3/2}}{3-2\alpha} \tag{10}$$

Fig.2 depicts the boundaries for various values of time t (in the moving system of coordinates). The dashed line represents the region of influence in uniform motion. Calculation shows that the angle of spread of the disturbance zone depends upon neither acceleration nor time, and coincides with the value of the angle in uniform motion. The region is now, however, bounded by a closed curve, changing with time and similar relative to the origin of coordinates, with the similarity coefficient $\frac{1}{2}at^2$.

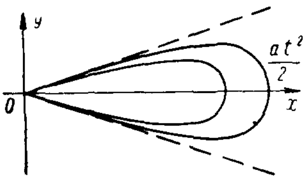

Fig. 2

Construction of curves of constant phase is the next step. The coordinates of points $P(x, y)$ at which the disturbance is to be calculated can be written in the form

$$x = OQ - r\cos\theta = at\tau - {}^1/_2 a\tau^2 - r\cos\theta, \qquad y = r\sin\theta$$

In view of condition (7), x and y cam be expressed in terms of r as follows

$$x = at\tau - \frac{a\tau^2}{2} - \frac{2r^2}{a\tau(t-\tau)}, \qquad y = \pm r\left(1 - \frac{4r^2}{a^2\tau^2(t-\tau)^2}\right)^{1/2}$$

Next is found the locus of points for which the phase $\varphi = {}^1/_4 g\tau^2 r^{-1} = \text{const}$.

Writing r in terms of φ leads to the equation of the curves of constant phase

$$x = at\tau - \frac{a\tau^2}{2} - \frac{g^2\tau^3}{8a\varphi^2(t-\tau)}, \qquad y = \pm\frac{g\tau^2}{4\varphi}\left[1 - \frac{g^2\tau^2}{4a^2\varphi^2(t-\tau)^2}\right]^{1/2}$$

These formulas can be simplified; introduction of the quantity $c = \frac{1}{2}g/a\varphi$ and the dimensionless ratio α leads to the result

$$x = \frac{at^2}{2}\,\frac{\alpha}{1-\alpha}\,[\alpha^2(1-c^2) - 3\alpha + 2], \quad y = \pm\frac{at^2}{2}\,\frac{c\alpha^2}{1-\alpha}\sqrt{\alpha^2(1-c^2) - 2\alpha + 1} \tag{11}$$

Inasmuch as the subradical expression in the above formula cannot be negative, limits to the variation in α are given by the inequality

$$0 \leqslant \alpha \leqslant (1+c)^{-1}.$$

Ascribing some fixed value to c and varying α within the above, limits results in a constant-phase curve. The shape of these curves is shown in

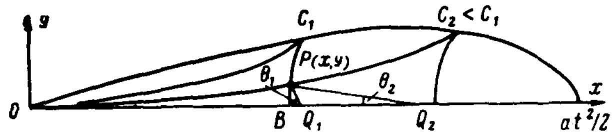

Fig. 3

Fig.3. Clearly visible are the systems of divergent and transverse waves, just as in uniform motion. From Equations (11) it follows

$$\begin{aligned}\frac{dx}{d\alpha} &= \frac{at^2}{2}\,\frac{[2\alpha^3(1-c^2) - 3\alpha^2(2-c^2) + 6\alpha - 2]}{(1-\alpha)^2}\\ \frac{dy}{d\alpha} &= \mp\frac{at^2}{2}\,\frac{c\alpha\,[2\alpha^3(1-c^2) - 3\alpha^2(2-c^2) + 6\alpha - 2]}{(1-\alpha)^2\sqrt{\alpha^2(1-c^2) - 2\alpha + 1}}\end{aligned} \tag{12}$$

The singular points of the curves correspond to the values $\alpha = \alpha_*$, for which

$$M(\alpha, c) = \alpha^3(1 - c^2) - 3\alpha^2(2 - c^2) + 6\alpha - 2 = 0 \qquad (13)$$

These points lie on the boundary of the region of disturbance. It is also apparent from the construction that the system of divergent waves is obtained as α varies in the interval $0 \leqslant \alpha < \alpha_*$, while transverse waves correspond to values of α within the limits $\alpha_* < \alpha \leqslant (1 + c)^{-1}$.

It can be shown that curves of constant phase are normal to lines drawn back to the corresponding influence points. Indeed, from Equations (12) it follows that

$$\frac{dy}{dx} = \mp \frac{c\alpha}{\sqrt{\alpha^2(1 - c^2) - 2\alpha + 1}} = -\frac{1}{\tan\theta} \qquad (14)$$

Here Equation (7) was used, which leads to the result

$$\tan\theta = \pm[\alpha^2(1 - c^2) - 2\alpha + 1]^{1/2}(c\alpha)^{-1}$$

Equation (14) shows that the curves of constant phase are indeed orthogonal to lines drawn back to the corresponding influence points (Fig.3). Since through each point lying within the region of disturbance (with the exception of those lying on the ship's course) there pass two curves of constant phase, each such point is associated with two influence points. Also worthy of note is that when $\alpha = (1 + c)^{-1}$, $x = OB$ (Fig.3), and it follows from Equation (11),

$$x = \frac{at^2}{2(1 + c)} = \frac{at^2}{2}\frac{2a\varphi}{g + 2a\varphi}$$

Now the length of the transverse waves can be found, since a change in phase of 2π corresonds to a wave length. Indeed,

$$\lambda = \frac{at^2}{2}\frac{4\pi a g}{(g + 2a\varphi)(g + 4\pi a + 2a\varphi)}$$

With increasing distance from the origin of coordinates, the phase φ increases from 0 to ∞, and hence the length of the transverse waves does not stay constant, as in uniform motion, but decreases from the value $2\pi a^2t^2/(g + 4\pi a)$ to zero. The velocity of propagation of the transverse waves, equal to $(g\lambda/2\pi)^{1/2}$, decreases commensurately from the value

$$at\ \ (g/(g + 4\pi a))^{1/2}$$

to zero, i.e. it reamains always less than the speed of the ship at any given instant.

Finally, by applying the method of stationary phase, the amplitude of the waves can be studied. For this it is necessary to determine φ, ψ and $d^2\varphi/d\tau^2$ for those values of r and θ that satisfy the stationary condition (7).

From Equations (5) and (6) and the relationship $\theta = \tan^{-1} y/(x_1 - x)$ (Fig.1) it follows that

$$\frac{d^2\varphi}{d\tau^2} = \frac{g}{2r}\left[1 + \frac{a\tau^2}{2r}\cos\theta - \frac{a^2(t - \tau)^2\tau^2\sin^2\theta}{2r^2}\right]$$

It is more convenient to express $d^2\varphi/d\tau^2$ in terms of the parameters α and c, by substituting the expression $\cos\theta = c\alpha(1 - \alpha)^{-1}$ and $r = \frac{1}{2}at^2c\alpha^2$; then

$$\frac{d^2\varphi}{d\tau^2} = \frac{g}{at^2}\frac{[2\alpha^3(1 - c^2) - 3\alpha^2(2 - c^2) + 6\alpha - 2]}{c^3\alpha^4(1 - \alpha)} \qquad (15)$$

It is apparent that $\varphi''(\alpha, c) < 0$ when $0 \leqslant \alpha < \alpha_*$ and $\varphi''(\alpha, c) > 0$ when $\alpha_* < \alpha \leqslant (1 + c)^{-1}$. The contribution of a point of stationary phase to the integral (3) is given by the following Formula [2 and 3]

$$\zeta(x, y, t) \sim \psi(\alpha, c)\left(\frac{2\pi}{|\varphi''(\alpha, c)|}\right)^{1/2}\sin\left[\varphi(\alpha, c) \pm \frac{\pi}{4}\right] \qquad (16)$$

Here, c and α are parameters characterizing the position of the point of stationary phase on the course of the ship for a given point $P(x, y)$. The sign of the additive constant $\pm \pi/4$ is taken in agreement with that of $\varphi''(\alpha, c)$. As was noted above, to each point within the region of disturbance there correspond two pairs of values of the parameters c and α which satisfy the stationary condition; one pair $(c_1; \alpha_1)$ pertains to the system of transverse waves $(\varphi''(\alpha_1; c_1) > 0,\ \alpha_* < \alpha_1 \leqslant (1+c)^{-1})$ while the second pair $(c_2; \alpha_2)$ refers to the system of divergent waves $(\varphi''(\alpha_2; c_2) < 0,\ 0 \leqslant \alpha_2 < \alpha_*)$. In view of the above, the relationship for the region within the zone of disturbance is found to be

$$\zeta(x, y, t) \sim -\frac{2Q\sqrt{g}}{\sqrt{\pi}\rho a^{1/2} t^4}\left[\frac{(1-\alpha_1)^{1/2}}{\alpha_1^3 c_1^{5/2} M^{1/2}(\alpha_1; c_1)} \exp\left(-\frac{2\nu g^2}{a^4 t^3 c_1^4 \alpha_1^3}\right) \sin\left(\frac{g}{2ac_1} + \frac{\pi}{4}\right) + \right.$$

$$\left. + \frac{(1-\alpha_2)^{1/2}}{\alpha_2^3 c_2^{5/2} M^{1/2}(\alpha_2; c_2)} \exp\left(-\frac{2\nu g^2}{a^4 t^3 c_2^4 \alpha_2^3}\right) \sin\left(\frac{g}{2ac_2} - \frac{\pi}{4}\right)\right] \tag{17}$$

From this formula it is seen that the two systems of waves differ in phase by $\pi/2$ at every point where $c_1 = c_2$. Passage from the c, α parameters to x, y coordinates is achieved by Expressions (12).

Equation (17) is not applicable on the boundary of the region, where $\varphi''(\alpha, c) = 0$. To determine the amplitude along the boundary it is necessary to calculate the derivative $\varphi'''(\alpha, c)$ for the condition $\varphi''(\alpha, c) = 0$. Without difficulty, it is found that

$$\frac{d^3\varphi}{d\tau^3} = \frac{6g}{at^3} \frac{(2-\alpha)\left[\alpha^2(1-c^2) - 2\alpha + 1\right]}{c^2\alpha^5(1-\alpha)} \tag{18}$$

The amplitude along the boundary is given by Expression [2 and 3]

$$\zeta(x, y, t) \sim \frac{\Gamma(1/3)}{\sqrt{3}} \psi(\alpha_*) \left(\frac{6}{|\varphi'''(\alpha_*)|}\right)^{1/3} \sin\varphi(\alpha_*) \tag{19}$$

Since α_* is a root of Equation (13), the parameter c can be eliminated, expressing it in terms of α, as follows

$$c^2 = \frac{2(1-\alpha)^3}{\alpha^2(3-2\alpha)} \tag{20}$$

Substitution of the expressions for φ, ψ and φ''' on the boundary of the region into Equation (19) and taking into account Equation (20), it is found that

$$\zeta(x, y, t) \sim \frac{Q\Gamma(1/3)\, g^{2/3}}{2\sqrt{3}\pi\rho a^{11/3} t^4} \frac{(3-2\alpha)^{11/6}}{\alpha^{1/3}(1-\alpha)^{31/6}(2-\alpha)^{1/3}} \times$$

$$\times \exp\left[-\frac{\nu g^2 \alpha(3-2\alpha)^2}{2a^4 t^3(1-\alpha)^6}\right] \sin\frac{g\alpha(3-2\alpha)^{1/2}}{2\sqrt{2}a(1-\alpha)^{3/2}} \qquad (0 \leqslant \alpha \leqslant 1) \tag{21}$$

Formulas (10) permit expression of the parameter α in terms of the x, y coordinates of points on the boundary of the region of disturbance.

As is well known, the method of stationary phase is applicable to rapidly fluctuating functions, and hence the approximation will be sufficiently valid if the phase in Equation (17) is sufficiently large, i.e. if $\frac{1}{2}g/ac$ is sufficiently large. Physically, this means that the approximation will be satisfactory at sufficiently large distances from the point impulse.

BIBLIOGRAPHY

1. Nikitin, A.K. and Podrezov, C.A., K prostranstvennoi zadache o volnakh na poverkhnosti viazkoi zhidkosti beskonechnoi glubiny (On the spatial problem of waves on the surface of a viscous fluid of infinite depth). *PMM* Vol.28, № 3, 1964.

2. Stoker, J.J., Volny na vode (Water Waves),(Russian translation). Izd.inostr.Lit., 1959 (*).

3. Sretenskii, L.N., Teoriia volnovykh dvizhenii zhidkosti (Theory of Wave Motion in Fluids). ONTI, M., 1936.

Translated by T.S.S.

Editorial Note

*) Stoker, J.J., Water Waves: The Mathematical Theory with Applications. Interscience Publishers, Inc., New York , 1957.